On the Relations between Eigenvalues and Eigenvectors for Matrices Resulting from Pre and Post Multiplication by the Transpose

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Abstract. Relationships between the eigenproblem associated with the matrices A^TA and AA^T are derived. If the larger problem is solved first, then the eigenvalues and eigenvectors associated with the smaller problem may easily be computed from the derived relationships.

Consider a nonsquare matrix A of rank m and of dimension $c \times d$. The product A^TA is symmetric and positive semidefinite, and hence possesses an eigenvector-eigenvalue relationship of the form

$$S^T(A^TA)S = D\,, \tag{1}$$

where S is a normal orthogonal matrix containing eigenvectors as columns, and D is a diagonal matrix containing eigenvalues as the diagonal elements. S and D, both of dimension $d \times d$, can be ordered so that any zero eigenvalues, resulting from A^TA having excess dimension over the rank m, are at the bottom:

$$D = \left[\begin{array}{c|c} E & 0 \\ \hline 0 & 0 \end{array}\right] \tag{2}$$

where E is an $m \times m$ submatrix with nonzero diagonal elements. Correspondingly, the ordered S can be partitioned as

$$S = [S_m \,\vdots\, S_p]\,, \tag{3}$$

so that the left-hand part S_m contains those eigenvectors associated with nonzero eigenvalues. Here $p = d - m$.

The positive semidefinite property of A^TA assures that the diagonal elements of E are positive, hence E can be "square-rooted":

$$E = E^{1/2}E^{1/2} \tag{4}$$

with $E^{1/2}$ being a diagonal matrix containing the square roots of the nonzero eigenvalues of A^TA.

The basic relationship (1) can be modified by premultiplication by S to give

$$(A^TA)S = SD\,. \tag{5}$$

If (5) is written in partitioned form, the following result becomes obviously valid:

$$A^TAS_m = S_mE\,. \tag{6}$$

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A normal orthogonal vector set can be constructed as follows: First substitute (4) into (6) and premultiply by S_m^T, then premultiply and postmultiply by $E^{-1/2}$ to give

$$(AS_mE^{-1/2})^T(AS_mE^{-1/2}) = I . \tag{7}$$

Equation (6) is a good starting point for investigating the eigenproblem of AA^T. Premultiplying (6) by A and postmultiplying that result by $E^{-1/2}$ gives

$$(AA^T)(AS_mE^{-1/2}) = AS_mE^{1/2} = (AS_mE^{-1/2})E . \tag{8}$$

Now, from (7) and (8) it follows that

$$(AS_mE^{-1/2})^T(AA^T)(AS_mE^{-1/2}) = E . \tag{9}$$

Now denote the eigenvector-eigenvalue relationship for AA^T similarly to (1) as

$$R^T(AA^T)R = D^* , \tag{10}$$

where R and D^* are of dimension $c \times c$. Reorder R and D^* to obtain

$$R = [R_m \mid R_q] \quad \text{and} \quad D^* = \left[\begin{array}{c|c} E^* & 0 \\ \hline 0 & 0 \end{array}\right] .$$

Here $q = c - m$ and E^* is an $m \times m$ diagonal submatrix.

The following properties are well known:

(1) $E^* = E$.

(2) R_q is arbitrary, subject to the restrictions $R_q^TR_m = 0$, $R_q^TR_q = I$.

The important result of the paper is that

$$R_m = AS_mE^{-1/2} \tag{11}$$

which follows from an examination of (9).

In conclusion, if the larger problem (i.e., the eigenproblem associated with A^TA) is solved first, then the complete eigenproblem associated with AA^T may be computed by using the above-mentioned well-known properties together with the result (11).

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